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Positive periodic solutions of second-order differential equations[☆]

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Abstract

In this work, we have obtained existence results for single and multiple positive periodic solutions for a class of second-order differential equations. Our results are based on a fixed point theorem for cones.

Keywords: Positive periodic solution; Fixed point theorem; Cone; Differential equation

1. Introduction

In this work, we consider the existence of positive periodic solutions of second-order differential equations

$$x''(t) + a(t)x'(t) + f(t, x(t)) = 0, \tag{1.1}$$

where $a(t) \in C(R, R),\ f(t, u) \in C(R \times R, R),\ a(t + \omega) = a(t),\ f(t + \omega, u) = f(t, u)$.

The existence problems for periodic solutions are always an important aspect of differential equation qualitative analysis. The existence problems for periodic solutions and periodic boundary value problems for first-order and second-order ordinary differential equations have attracted the attention of many authors. Many theorems and methods of nonlinear functional analysis have been applied to these problems. These theorems and methods are mainly the upper and lower solutions method and the monotone iterative technique (see [2,8,12]), the continuation method of topological degree (see [3,9,10,13]), fixed point theorems, the variational method and critical point theory (see [4,6,7,11]). However, relatively few papers have discussed the existence of positive periodic solutions for second-order differential equations. Li [11] has discussed the existence of positive periodic solutions of differential equations

$$x''(t) + g(t, x(t)) = 0, \tag{1.2}$$

where $g(t, u) \in C(R \times R, R)$, $g(t + \omega, u) = g(t, u)$. By using the theory of the fixed point index for cones, the author has obtained the optimal conditions for the nonlinear term g so that Eq. (1.2) has a positive periodic solution. But this method is not valid for Eq. (1.1) with the term x'. By using a new fixed point theorem for cones, we obtain existence results for single and multiple positive periodic solutions to Eq. (1.1) which extend the results given in [11].

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2. Some preparation

Let

$$C_\omega = \{u \in C(R,R) : u(t+\omega) = u(t)\}, \qquad C^1_\omega = \{u \in C^1(R,R) : u(t+\omega) = u(t)\},$$
$$M_1 = \left\{u \in C^1_\omega : \int_0^\omega u(t) > 0\right\}, \qquad M_2 = \left\{u \in C_\omega : \int_0^\omega u(t) > 0\right\},$$
$$M_3 = \{u \in C_\omega : u(t) > 0\}.$$

Clearly, C_ω is a Banach space, equipped with the norm $\|u\| = \max_{0\le t\le\omega} |u(t)|$.
For convenience, we list the following assumptions:
(H_1) $a = \alpha+\beta, \alpha \in M_1, \beta \in M_2, r_1 = \alpha\beta+\alpha' \in M_3,\ F_1 = (\alpha\beta+\alpha')v - f(t,v) \ge 0$ for $t \in [0,\omega]$ and $v \ge 0$.
(H_2) $a = \alpha-\beta, \alpha \in M_1, \beta \in M_2, r_2 = \alpha\beta-\alpha' \in M_3,\ F_2 = (\alpha\beta-\alpha')v + f(t,v) \ge 0$ for $t \in [0,\omega]$ and $v \ge 0$.
(H_3) $a = -\alpha-\beta, \alpha \in M_1, \beta \in M_2, r_3 = \alpha\beta-\alpha' \in M_3,\ F_3 = (\alpha\beta-\alpha')v - f(t,v) \ge 0$ for $t \in [0,\omega]$ and $v \ge 0$.
(H_4) $a = -\alpha+\beta, \alpha \in M_1, \beta \in M_2, r_4 = \alpha\beta+\alpha' \in M_3,\ F_4 = (\alpha\beta+\alpha')v + f(t,v) \ge 0$ for $t \in [0,\omega]$ and $v \ge 0$.
Let $p, q \in C_\omega$ and consider the following two differential equations:

$$u'(t) = -p(t)u(t) + q(t), \tag{2.1}$$
$$u'(t) = p(t)u(t) - q(t). \tag{2.2}$$

Lemma 2.1. *Assume that $p \in M_2$, then Eq. (2.1) has a unique ω-periodic solution*

$$u(t) = \int_t^{t+\omega} \frac{\exp\int_t^s p(r)\mathrm{d}r}{\exp\left(\int_0^\omega p(r)\mathrm{d}r\right)-1} q(s)\mathrm{d}s,$$

and Eq. (2.2) has a unique ω-periodic solution

$$u(t) = \int_t^{t+\omega} \frac{\exp\int_s^{t+\omega} p(r)\mathrm{d}r}{\exp\left(\int_0^\omega p(r)\mathrm{d}r\right)-1} q(s)\mathrm{d}s.$$

Put

$$A_\eta(t,s) = \frac{\exp\int_t^s \eta(r)\mathrm{d}r}{\exp\left(\int_0^\omega \eta(r)\mathrm{d}r\right)-1}, \qquad \widetilde{A}_\eta(t,s) = \frac{\exp\int_s^{t+\omega} \eta(r)\mathrm{d}r}{\exp\left(\int_0^\omega \eta(r)\mathrm{d}r\right)-1}\mathrm{d}r,$$
$$G^1_\alpha(t,s) = G^2_\alpha(t,s) = A_\alpha(t,s), \qquad G^3_\alpha(t,s) = G^4_\alpha(t,s) = \widetilde{A}_\alpha(t,s),$$
$$G^1_\beta(t,s) = G^4_\beta(t,s) = A_\beta(t,s), \qquad G^2_\beta(t,s) = G^3_\beta(t,s) = \widetilde{A}_\beta(t,s),$$
$$m = \frac{1}{\exp\left(\int_0^\omega \alpha(r)\mathrm{d}r\right)-1}, \qquad M = \frac{\exp\int_0^\omega \alpha(r)\mathrm{d}r}{\exp\left(\int_0^\omega \alpha(r)\mathrm{d}r\right)-1}.$$

Next, we define a mapping $T^i : C_\omega \to C_\omega$ by

$$(T^i x)(t) = \int_t^{t+\omega} G^i_\alpha(t,s) \int_s^{s+\omega} G^i_\beta(s,u) F_i(u, x(u))\mathrm{d}u\mathrm{d}s, \quad i = 1,2,3,4. \tag{2.3}$$

and a cone K in C_ω by

$$K = \{u \in C_\omega : u(t) \ge \delta\|u\|,\ \forall t \in R\}$$

where $\delta = m/M \in (0,1)$.
The proof of the following lemma is easy; we omit it.

Lemma 2.2. *Assume that (H_i) $(i = 1,2,3,4)$ is satisfied, then the fixed point of F^i is a solution of Eq. (1.1).*

Lemma 2.3. *Assume that (H_i) $(i = 1,2,3,4)$ is satisfied, then $T^i(K) \subset K$ and $T^i : K \to K$ is completely continuous.*

Proof. We only prove that the (H_1) case is satisfied. If (H_1) holds and $x \in K$, then $F_1(\cdot, x(\cdot)) \in C_\omega$. It follows from $\beta \in M_2$ and Lemma 2.1 that $\int_s^{s+\omega} G_\beta^1(s,u)F_1(u,x(u))\mathrm{d}u \in C_\omega$. Further we have $(T^1x)(t) \in C_\omega$.

On the other hand, we have

$$\|T^1x\| \le M \int_t^{t+\omega}\int_s^{s+\omega} G_\beta^1(s,u)F_1(u,x(u))\mathrm{d}u\mathrm{d}s,$$
$$(T^1x)(t) \ge m \int_t^{t+\omega}\int_s^{s+\omega} G_\beta^1(s,u)F_1(u,x(u))\mathrm{d}u\mathrm{d}s.$$

Hence

$$(T^1x)(t) \ge \frac{m}{M}\|T^1x\|,$$

which implies $T^1x \in K$. Thus $T^1(K) \subset K$.

Obviously, $T^1 : K \to K$ is continuous. Let $D \subset K$ be a bounded set. For every $x \in D$, it is easy to see that there exists a constant $d > 0$ such that $\|T^1x\| \le d$. And

$$(T^1x)'(t) = -\alpha(t)(T^1x)(t) + \int_t^{t+\omega} G_\beta^1(t,u)F_1(u,x(u))\mathrm{d}u.$$

It follows that $\{(F^1x)' \mid x \in D\}$ is a bounded set. Consequently, $T^1(D)$ is an equicontinuous and bounded family of functions. Thus, by the Arzela–Ascoli theorem, $T^1 : K \to K$ is completely continuous. The proof is complete. □

To conclude this section, we state a new fixed point theorem for cones which will be needed in this work.

Lemma 2.4 ([1,5]). *Let X be a Banach space and K be a cone in X. Suppose Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$ and suppose that*

$$\Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

is a completely continuous operator such that

(i) $\|\Phi, u\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and there exists* $\psi \in K \setminus \{0\}$ *such that* $x \neq Tx + \lambda\psi$ *for* $x \in K \cap \partial\Omega_2$ *and* $\lambda > 0$, *or*

(ii) $\|\Phi, u\| \le \|u\|$ *for* $u \in K \cap \partial\Omega_2$, *and there exists* $\psi \in K \setminus \{0\}$ *such that* $x \neq Tx + \lambda\psi$ *for* $x \in K \cap \partial\Omega_2$ *and* $\lambda > 0$.

Then Φ has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3. Main results

Put

$$\phi_i(s) = \min\left\{\frac{F_i(t,u)}{r_i(t)u}, t \in [0,\omega], u \in [\delta s, s]\right\},$$
$$\varphi_i(s) = \max\left\{\frac{F_i(t,u)}{r_i(t)u}, t \in [0,\omega], u \in [\delta s, s]\right\},$$
$$\phi_i^0 = \lim_{s\to 0^+}\phi_i(s), \qquad \phi_i^\infty = \lim_{s\to\infty}\phi_i(s),$$
$$\varphi_i^0 = \lim_{s\to 0^+}\varphi_i(s), \qquad \varphi_i^\infty = \lim_{s\to\infty}\varphi_i(s).$$

Theorem 3.1. *Assume that (H_i) holds, and there exist two positive constants $a,\ b : a \neq b$ such that $\varphi_i(a) \le 1$ and $\phi_i(b) \ge 1$; then Eq. (1.1) has at least a positive periodic solution $x \in K$ with $\min\{a,b\} \le \|x\| \le \max\{a,b\}$.*

Proof. Without loss of generality, we assume that $a < b$. Let $\Omega_1 = \{x \in K, \|x\| < a\}$ and $\Omega_2 = \{x \in K, \|x\| < b\}$. By $\varphi_i(a) \le 1$ and $\phi_i(b) \ge 1$, we have

$$F_i(t,x) \le r_i(t)x, \quad \forall\, 0 \le t \le \omega,\ \delta a \le x \le a,$$
$$F_i(t,x) \ge r_i(t)x, \quad \forall\, 0 \le t \le \omega,\ \delta b \le x \le b.$$

Next, we show that:
(i) $\|T^1 x\| \le \|x\|$ for $x \in K \cap \partial \Omega_1$.
(ii) There exists $\psi \in K \setminus \{0\}$ such that $x \neq Tx + \lambda\psi$ for $x \in K \cap \partial\Omega_2$ and $\lambda > 0$.
To justify (i), let $x \in K \cap \partial\Omega_1$, then $\|x\| = a$, $\delta a \le x(t) \le a$. So we have that

$$\begin{aligned}(T^i x)(t) &= \int_t^{t+\omega} G_\alpha^i(t,s)\int_s^{s+\omega} G_\beta^i(s,u)F_i(u,x(u))\mathrm{d}u\mathrm{d}s\\ &\le \int_t^{t+\omega} G_\alpha^i(t,s)\int_s^{s+\omega} G_\beta^i(s,u)r_i(u)x(u)\mathrm{d}u\mathrm{d}s\\ &\le \|x\|\int_t^{t+\omega} G_\alpha^i(t,s)\int_s^{s+\omega} G_\beta^i(s,u)r_i(u)\mathrm{d}u\mathrm{d}s\\ &= \|x\|\end{aligned}$$

and this implies $\|T^i x\| \le \|x\|$, $x \in K \cap \partial\Omega_1$.
Next, to prove (ii). Let $\psi \equiv 1$. Assume that there exists $x \in K \cap \partial\Omega_2$, and $\lambda > 0$ such that

$$x = T^i x + \lambda\psi.$$

Clearly $\delta b \le x(t) \le b$. Let $\chi = \min\{x(t), t \in [0,\omega]\}$, then we have $\chi = x(t)$ for some $t \in [0,\omega]$. Thus it follows that

$$\begin{aligned}x(t) &= (T^i x)(t) + \lambda\\ &= \int_t^{t+\omega} G_\alpha^i(t,s)\int_s^{s+\omega} G_\beta^i(s,u)F_i(u,x(u))\mathrm{d}u\mathrm{d}s + \lambda\\ &\ge \int_t^{t+\omega} G_\alpha^i(t,s)\int_s^{s+\omega} G_\beta^i(s,u)r_i(u)x(u)\mathrm{d}u\mathrm{d}s + \lambda\\ &\ge \chi\int_t^{t+\omega} G_\alpha^i(t,s)\int_s^{s+\omega} G_\beta^i(s,u)r_i(u)\mathrm{d}u\mathrm{d}s + \lambda\\ &= \chi + \lambda\end{aligned}$$

and this implies $\chi > \chi$, a contradiction.
By Lemma 2.4, it follows that F^i has a fixed point $x \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. Furthermore, $a \le \|x\| \le b$ and $x(t) \ge \delta a > 0$, which means that $x(t)$ is an ω-periodic positive solution of Eq. (1.1). The proof is complete. □

Corollary 3.1. *Assume that (H_i) holds. If one of the following conditions holds:*
(i) $\varphi_i^0 < 1,\ \phi_i^\infty > 1$,
(ii) $\phi_i^0 > 1,\ \varphi_i^\infty < 1$,
then Eq. (1.1) has at least a positive ω-periodic solution.

Remark. Put $\alpha = \beta = c$ (c is a positive constant) and $i = 2$ or $i = 4$; Corollary 3.1 is Theorem 1.1 in [11].

Theorem 3.2. *Assume that (H_i) is satisfied, and there exist $N+1$ positive constants $p_1 < p_2 < \cdots < p_N < p_{N+1}$ such that one of the following conditions is satisfied:*
(i) $\varphi_i(p_{2k-1}) < 1,\ k = 1,2,\ldots,[(N+2)/2]$, $\phi_i(p_{2k}) > 1,\ k = 1,2,\ldots,[(N+1)/2]$,
(ii) $\phi_i(p_{2k-1}) > 1,\ k = 1,2,\ldots,[(N+2)/2]$, $\varphi_i(p_{2k}) < 1,\ k = 1,2,\ldots,[(N+1)/2]$,
where $[d]$ denotes the integer part of d. Then, Eq. (1.1) has at least N positive periodic solutions $x_j \in K$, $j = 1,2,\ldots,N$ with $p_j < \|x_j\| < p_{j+1}$.

Proof. It is enough to prove the case (i). Since $\varphi, \phi : (0,\infty) \to (0,\infty)$ are continuous, it follows that there exist $a_k,\ b_k : p_k < a_k < b_k < p_{k+1},\ k = 1,2,\ldots,N$ such that

$$\begin{aligned}&\varphi_i(a_{2k-1}) \le 1, \qquad \phi_i(b_{2k-1}) \ge 1, \quad k = 1,2,\ldots,[(N+1)/2],\\ &\phi_i(a_{2k}) \ge 1, \qquad \varphi_i(b_{2k}) \le 1, \quad k = 1,2,\ldots,[(N+1)/2].\end{aligned}$$

It follows from Theorem 3.1 that Eq. (1.1) has at least a periodic solution $x_j \in K$ with $a_j \le \|x_j\| \le b_j$ for the pairs (a_j, b_j), $j = 1,2,\ldots,N$. The proof is complete. □

Corollary 3.2. *Assume that* (H_i) *is satisfied, and*

$$\varphi_i^0 < 1, \qquad \varphi_i^\infty < 1.$$

Further there is a positive constant b such that $\phi_i(b) > 1$. *Then Eq.* (1.1) *has at least two positive* ω*-periodic solutions* x_1 *and* x_2 *such that*

$$0 < \|x_1\| < b < \|x_2\|.$$

Corollary 3.3. *Assume that* (H_i) *is satisfied, and*

$$\phi_i^0 > 1, \qquad \phi_i^\infty > 1.$$

Further there is a positive constant a such that $\varphi_i(a) < 1$. *Then Eq.* (1.1) *has at least two positive* ω*-periodic solutions* x_1 *and* x_2 *such that*

$$0 < \|x_1\| < a < \|x_2\|.$$

4. Some examples

In this section, we apply the main results obtained in previous sections to several examples.

Example 4.1. Consider the differential equation

$$x''(t) + x'(t)\sin t + \mathrm{e}^{-x(t)} - (2+\sin t)x(t) = 0. \tag{4.1}$$

Let $\sin t = \beta - \alpha$, $\alpha(t) = 2$, $\beta(t) = 2 + \sin t$. Then $F_4(t,u) = \mathrm{e}^{-u} + (2+\sin t)u$ and $r_4 = 2(2+\sin t) > 0$.

Note that $\varphi_4^0 = \infty$, $\phi_4^\infty = \frac{1}{2}$. Eq. (4.1) has at least a positive 2π-periodic solution from Corollary 3.1.

Example 4.2. Consider the differential equation

$$x''(t) + x^2(t) - 3x(t) + \frac{1}{5}\sin^2 4\pi t + 1 = 0. \tag{4.2}$$

Eq. (4.2) has at least two positive periodic solutions. In fact, let $\alpha(t) = \beta(t) = 1$, then $\delta = \mathrm{e}^{-\frac{1}{2}}$, $F_2(t,u) = u^2 + 1 - 2u + \frac{1}{5}\sin^2 4\pi t$.

Note that $\phi_2^0 = \infty$, $\phi_2^\infty = \infty$. Let $a = \frac{5}{2}$, then for $\delta a \le u \le a$, $F_2(t,u) = u^2 + 1 - 2u + \frac{1}{5}\sin^2 4\pi t < u$. By Corollary 3.3, Eq. (4.2) has at least two positive periodic solutions.

Example 4.3. Consider the differential equation

$$x''(t) + 2x'(t) + \sin x(t) - \frac{1}{100}(1 + \sin 200\pi t) = 0. \tag{4.3}$$

Put $\alpha = \beta = 1$, then $\delta = \mathrm{e}^{-\frac{1}{100}}$, $F_1(t,u) = u - \sin u + \frac{1}{100}(1 + \sin 200\pi t)$. Let $p_{2k-1} = 2k\pi + \frac{2\pi}{3}$, $p_{2k} = 2(k+1)\pi - \frac{2\pi}{3}$, $k = 1, 2, \ldots$. If $\delta p_{2k-1} \le u \le p_{2k-1}$, then $\varphi_1(p_{2k-1}) < 1$. If $\delta p_{2k} \le u \le p_{2k}$, then $\phi_1(p_{2k}) > 1$.

By Theorem 3.2, Eq. (4.3) has infinite positive periodic solutions.

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